# FLOW EQUATION AND ITS APPLICATION TO WATER MOVEMENT IN SOILS

by

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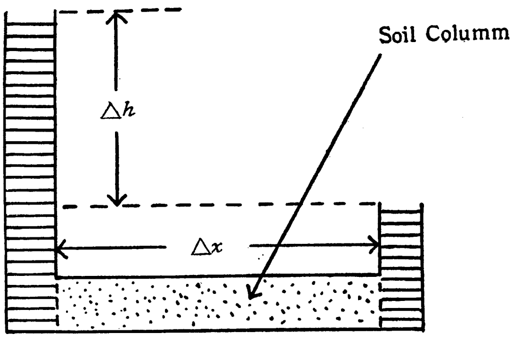
## I. EVOLUTION OF FLOW EQUATION

Water always moves from higher potential site to lower potential site. The basic equation describing the relation between the velocity of flow and the potential difference is given by Darcy's law, which is based on the experimental observation. The simplest example leading to Darcy's equation is given in the follwing. In this case, it is one dimensional saturated fiow.




Darcy's equation is given by:

$$\frac{1}{A}\cdot\frac{\triangle Q}{\triangle t}=-K_1\frac{\triangle h}{\triangle x} \quad \text{or} \quad \frac{1}{A}\cdot\frac{\triangle Q}{\triangle t}=-K_2\frac{\triangle \phi}{\triangle x}$$

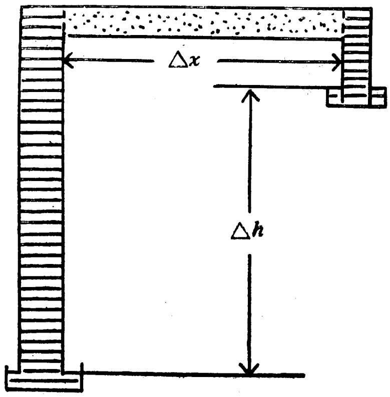
where A is cross sectional area, $\triangle Q$ is amount of water which passed the soil column in time $\triangle t$, usually measured by volume, $\triangle t$ is time in second, $\triangle h$ is difference in hydrolic head, $K_1$ and $K_2$ are conductivity, they are constants if soil is uniform. $\phi$ is potential, in this case $\phi=\rho gh$, $\triangle x$ is length of soil column, $\rho$ is density of water and $g$ is gravitational acceleration.

In differential form:

$$\frac{1}{A}\cdot\frac{dQ}{dt}=-K_1\frac{dh}{dx} \quad \text{or} \quad \frac{1}{A}\cdot\frac{dQ}{dt}=-K_2\frac{d\phi}{dx}$$

The negative sign comes from the fact that if we take $x$ as increasing from left to right, and water flow as positive when it flows from left to right, since $\frac{\triangle h}{\triangle x}$ is always negative when flow is positive, this is a necessary consequence.

In case of unsaturated flow, we may change our experimental set up to the following:




However, there are two states in unsaturated flow. One is steady state, the other is transient state. If sufficient time has elapsed, so that the moisture distribution pattern in the soil column does not change any more, the flow under this condition we call it steady state. In other words, the flow rate is everywhere the same along the soil column. On the other hand, if the moisture distribution pattern is changing, while the flow is taking place, we call it transient state. In other words, the flow rate is no longer constant along soil column.

In case of steady state, Darcy's equation can be writen as follows:

$$\frac{1}{A}\cdot\frac{\Delta Q}{\Delta t}=-\overline{K(\phi)}\frac{\Delta\phi}{\Delta x} \quad \text{where} \quad \overline{K(\phi)}=\frac{\int_{\phi_1}^{\phi_2}Kd\phi}{\phi_2-\phi_1}$$

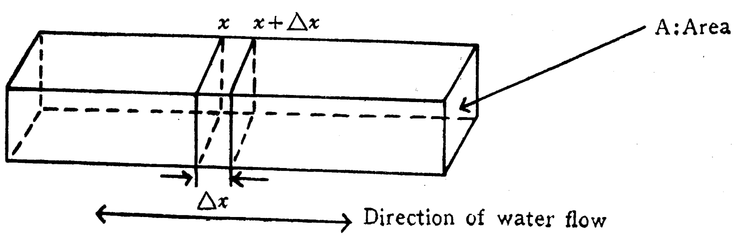
This situation makes the above experimental set up pretty hard to show whether the Darcy's law holds in unsaturated flow or not.

Darcy's equation when applied to unsaturated flow is best written in partial differential equation.

$$\frac{1}{A}\cdot\left(\frac{\partial Q}{\partial t}\right)_x=-K(\phi)\left(\frac{\partial\phi}{\partial x}\right)_t$$

Where $\left(\frac{\partial Q}{\partial t}\right)_t$ is rate of water flow across the cross section at $x$ and is function os time, $\left(\frac{\partial\phi}{\partial x}\right)_t$ is potential gradient at $t$ and is function of $x$, $K(\phi)$ is conductivity, it shows that K is either function of potential or moisture content, and not a constant as in saturated flow. This equation can be applied in both steady and transient state.

Our next step is to find a partial differential equation which describe the change of moisture distribution pattern with time based on the Darcy's law. For the sake of simpicity, let us consider horizontal flow.



First of all, we have to find the net increase of water in a small volume element $A\triangle x$. If we use the same principle that water flow is positive when it flows to right and $x$ is increasing as it goes to the right, the net increase of water is the difference between the water which passed through the cross section at $x$ and at $x+\triangle x$. In mathematical form:

$$A\triangle x\triangle\theta=\triangle Q_x-\triangle Q_{x+\triangle y}=\left[-AK(\phi)\frac{\partial\phi}{\partial x}\right]_x\triangle t-\left[-AK(\phi)\frac{\partial\phi}{\partial x}\right]\triangle t_{u+\triangle x\theta}$$

where $\theta$ is moisture content, and the others as before.

After transposition

$$\frac{\triangle\theta}{\triangle t}=\frac{K(\phi)\frac{\partial\phi}{\partial x}\Big|_{x+\triangle x}-K(\phi)\frac{\partial\phi}{\partial x}\Big|_x}{\triangle x}$$

In limiting case, it becomes:

$$\left(\frac{\partial\theta}{\partial t}\right)_x=\frac{\partial}{\partial x}\left[K(\phi)\frac{\partial\phi}{\partial x}\right]_t$$

If we introduce $\frac{\partial\phi}{\partial x}=\frac{d\phi}{d\theta}\cdot\frac{\partial\theta}{\partial x}$ we can eliminate $\phi$ so that result is:

$$\left(\frac{\partial\theta}{\partial t}\right)_x=\frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right]_t$$

wher $D(\theta)=K(\phi)\frac{d\phi}{d\theta}$, and is called diffusivity which usually considered as function of $\theta$ only.

In case of three dimension, it is only a matter of methematical manupulation to show that the net increase in a given volume element $\triangle x\triangle y\triangle z$ is the sum of net increase coming from the flows in X, Y and Z directions. This gives:

$$\left(\frac{\partial\theta}{\partial t}\right)_{x,y,z}=\frac{\partial}{\partial x}\left[K(\phi)\frac{\partial\phi}{\partial x}\right]_{t,y,z}+\frac{\partial}{\partial y}\left[K(\phi)\frac{\partial\phi}{\partial y}\right]_{t,x,z}+\frac{\partial}{\partial x}\left[K(\phi)\frac{\partial\phi}{\partial z}\right]_{,x,y}$$

Using the vector notation

$$\frac{\partial\theta}{\partial t}=\nabla\cdot(K\nabla\phi)$$

where

$$\nabla=i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}$$

This is a general flow equation we are looking for.

## II. SOIUTION OF FIOW EQUATION IN SOME SPECIAL CASES

Theoretically, if we can find such function $\theta(x, y, z, t)$ which satisfy the flow equation and also the given boundary and initial conditions, our problem is solved. As a matter of fact, the situation is not as simple as I mentioned here. First of all, $D(\theta)$ so defined is not a constant, but rather a complicated function of $\theta$. This makes the differential equation hard to be solved even in its simplest case.

e. g.

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial\theta}{\partial x}\right)\quad\text{becomes}\quad\frac{\partial\theta}{\partial t}=\frac{dD}{d\theta}\left(\frac{\partial\theta}{\partial x}\right)^2+D\frac{\partial^2\theta}{\partial x^2}$$

is a non-linear form.

Howerver, in the following special cases, some useful knowledge still can be obtained by digging in this equation.

Case 1). Saturated flow.

In this case, K becomes constant and also $\frac{\partial\theta}{\partial t}$ becomes zero. This makes the general flow equation reduces to Laplace equation:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Case 2). Horizontal unsaturated steady state flow.

In this case $\frac{\partial\theta}{\partial t}=0$, so the flow equation becomes:

$$\frac{dD}{d\theta}\left(\frac{d\theta}{dx}\right)^2+D\frac{d^2\theta}{dx^2}=0$$

After integration we get

$$\ln\frac{D}{D_0}=-\int_{\theta_0}^{\theta}\frac{\theta''}{(\theta')^2}d\theta$$

where

$$\theta''=\frac{d^2\theta}{dx^2},\quad \theta'=\frac{d\theta}{dx}$$

$D_0$ is diffusivity at $\theta_0$

Since $\theta''$ and $\theta'$ can be obtained experimentally, this provides one method of evaluating D in unsaturated flow.

Same technique can be used in case of vertical flow, however, the result is more complicated. Conversely, if D is a known function of $\theta$, $\theta$ can be obtained as a function of $x$, since:

$$\frac{d}{dx}\left(D\frac{d\theta}{dx}\right)=0,\quad D\frac{d\theta}{dx}=C_1\quad\text{and}\quad\int Dd\theta=C_1x+C_2$$

$C_1$ and $C_2$ can be evaluated from the given boundary conditions.

Case 3). Horizontal unsaturated flow with the following boundary and initial conditions.

1). Soil column is semi-infintive with uniform initial $\theta_i$ at $t=0$.

2). At $t>0$ $\theta=\theta_0$ at $x=0$ and $\theta\to\theta_i$ as $x\to\infty$

This situation is handled by Boltzmann transformation which involves the following theorem.

Given a certain partial differential equation in $\theta$, $x$ and $t$. If by substitution of $\eta(x, t)$, $x$ and $t$ can be eliminated from the equation, and also the boundary and initial conditions can be described in terms of $\eta$ only, the solution satisfying the differential equation and also the boundary and initial conditions is function of $\eta$ only.

Here,

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial\theta}{\partial x}\right)\tag{1}$$

or

$$\left(\frac{\partial\theta}{\partial\eta}\right)_x\left(\frac{\partial\eta}{\partial t}\right)_x=\left[\frac{\partial}{\partial\eta}\left(D\frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}\right)\frac{\partial\eta}{\partial x}\right]_t \tag{2}$$

where $\eta$ is function of $x$ and $t$. If by substitution

$$\eta=xt^{-\frac{1}{2}} \quad \text{or} \quad \left(\frac{\partial\eta}{\partial t}\right)_x=-\frac{1}{2}xt^{-\frac{3}{2}} \quad \text{and} \quad \left(\frac{\partial\eta}{\partial x}\right)_t=t^{-\frac{1}{2}}$$

to (2), equation (1) reduce to

$$-\frac{1}{2}\eta\left(\frac{\partial\theta}{\partial\eta}\right)_x=\left[\frac{\partial}{\partial\eta}\left(D\frac{\partial\theta}{\partial\eta}\right)\right]_t \tag{3}$$

The corresponding boundary and initial conditions are given by

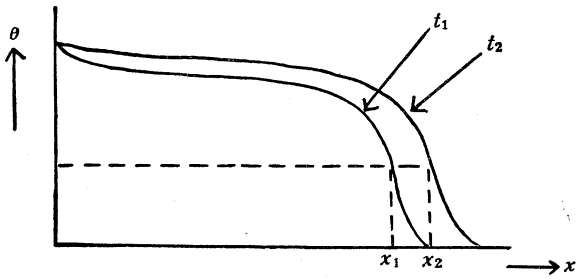
$$\eta\to 0 \quad \text{then} \quad \theta\to\theta_0, \quad \text{and} \quad \eta\to\infty \quad \text{then} \quad \theta\to\theta_i$$

where $\theta_0$ is the constant moisture content at $x=0$ after $t>0$, and $\theta_i$ is the initial moisture content through out the semi-infinitive soil column at $t=0$. Therefore, (3) can be written as ordinary differential equation.

$$-\frac{1}{2}\eta\frac{d\theta}{d\eta}=\frac{d}{d\eta}\left(D\frac{d\theta}{d\eta}\right)$$

Two important properties result from the above fact.

Property 1). Since $\theta$ is function of $\eta$ only in this case. If one solution is obtained at certain time experimentally, the entire solution is obtained consequently. In other words, from a certain moisture distribution pattern at certain time, we can construct any moisture distrbution pattern at any time.




In the above diagram, two line represent two moisture distribution at $t_1$ and $t_2$ respectively. If we draw a hoaizontal line which intersects these two lines at $x_1$ and $x_2$, the following equation holds:

$$\frac{x_1}{x_2}=\frac{\sqrt{t_1}}{\sqrt{t_2}}$$

So it is easy to find $x_2$ if the other three are known.

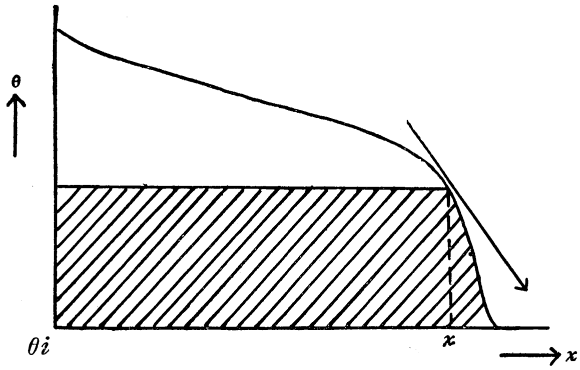
Property 2). From $-\frac{1}{2}\eta\frac{d\theta}{d\eta}=\frac{d}{d\eta}\left(D\frac{d\theta}{d\eta}\right)$ taking $t$ as constant substituting $d\eta=t^{-\frac{1}{2}}dx$ we get:

$$-\frac{1}{2}\frac{x}{t}\cdot\frac{d\theta}{dx}=\frac{d}{dx}\left(D\frac{d\theta}{dx}\right)$$

After integration and transposition we get:

$$D(\theta_x)=-\frac{1}{2t}\left[\frac{dx}{d\theta}\right]_{\theta_x}\cdot\int_{\theta_i}^{\theta_x}x\cdot d\theta \quad \left(\because \frac{d\theta}{dx}\Big|_{\theta_i}=0\right)$$

Diagramatically $\int_{\theta_i}^{\theta_x} x \cdot d\theta =$ shaded area $\frac{d\theta}{dx}$ = slope of the tangential line at $x$.




This provides another way of evaluating D experimentally, and Since this is not a steady state, it is easier to set up experimentally.

Case 4). Vertical unsaturated fiow with the same boundary and intial conditions as in case 3.

In this case, the flow equation becomes:

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right)+\frac{\partial K}{\partial z}$$

Since Boltzmann transformation never reduce this equation to ordinary differential equation, Philip (3) described another method of changing this equation to ordinary differential equation, and applied it to his "Theory of Infiltration".

Case 5). Vertical unsaturated steady state flow.

In this case the fiow equation becomes:

$$\frac{d}{dz}\left(D\frac{d\theta}{dz}\right)+\frac{dK}{dz}=0 \qquad \left(\because \frac{\partial\theta}{\partial t}=0\right)$$

If D and K are known function of $\theta$, it can be integrated immediately giving:

$$\int\frac{D}{C_1-K}d\theta=z+C_2$$

But it is not easy to find the value of $C_1$ and $C_2$ to fit the boundary conditions. Gardner (1) treated this case.

## REFERENCE


(1) Gardner, W. R. Some steady state solutions of the unsaturated flow equation with application to evaporation from a water table. Soil. Sci. 85:228, 1958.

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(3) Philip, J. R. The theory of infiltration: 2. The profile at infinty. Soil Sci. 83:435-448, 1957.

(4) Wylie, C. T. Advance Engineering Mathematics. McGraw-Hill Eook Company, Inc. 206-208.

# 流動方程式及其在土壤水分移動之應用

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## 摘　要

本文敍述流動方程式 (Flow Equation) 的誘導及其在土壤水分移動之應用。在本文所舉之例子可解該方程式而知水分移動之情形。