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三維暫態土壤水流模式顯式差分解之一致性及穩定性數值結構分析

The Analysis of Stability and Consistency for Solving Three Dimensional Transient Transport Equation of Soil Water Using Explicit Finite Difference Scheme

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ABSTRACT

The three dimensional numerical techniques being considered necessarily to analyse the slope stability problems, and that water content being evidently taken the greatest effects on soil strngth, this study was to clarify the numeric method of explicit time with central finite difference scheme for solving the governing equation of water movement in soils under the three dimenional unsteady state conditions.

After thorough analysis, in which exponential type of functions for hydraulic conductivity and soil water characteristic curve were incooperated as an example, the consistency of the governing equation was proved and its stability condition was defined. The stability condition is valid only when the gradient of matric potential in a solum is restricted.

Key Words : 3 − D, Soil water, stability, Consistency, Convergency.

摘　要

三維暫態水份流動模式之數值分析，因三維水份流動之研究，特別是用於邊坡穩定分析漸受重視而益形重要。本文首先說明該模式之導引，然後配合指數形式之水力導水度函數與水份特性函數之引用，分析顯式中央差分法用於該模式之穩定性、一致性及收斂性。最後獲得一致性、穩定性及收斂性之數值結構，但亦指出其穩定條件有其適用上限。

關鍵詞：三維，土壤水份，穩定性，一致性，收斂性。

前　言

近年坡地土石流及邊坡崩塌問題嚴重，有關邊坡受破壞機制之研究有進一步加強之必要。

邊坡破壞機制受地質材料性質、地質構造、地形效應、地震及氣候等因素所影響（賴與江，1985），而氣候因素中則以水

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份扮演最重要的角色。根據資料顯示，台灣山區崩坍頻率與年降雨量有密切關係(洪，1985)。究其原因爲降雨入滲造成土層吸水後自動增加、地下水位升高（黃，1985)、滲流應力增大 (蔡與蘇，19896)、有效應力之減少折損內磨擦效應；抑或是水份軟化土壤或岩石並改變其物化特性 (陳，1985 ；陳，1988 ；Mitchell ，1976) 等因素。故研究降雨入滲後之土壤水人分移動狀態當屬必然。

由於一般在探討邊坡穩定時，大抵皆將三維現況簡化爲二維切片分析，故而忽略切片兩翼邊力 (side forces) 之影響。但是學者早已指出，此等被忽略之邊際效應將可能造成35 %以上的估算誤差 (Wright ，1969)。因而三維邊坡穩定分析法一直爲學者們正視的課題 (Hovland, 1977; Chen and Chameau, 1982; Leshchinsky et al., 1985; Lam and Fredlund, 1993)。然而受限於方法的複雜性與穩定性，特別是結合不飽和土層與地下水變化過程，而以三維動態模式研究邊坡土石流的進展較少，其原因之一在於三維土體水文模擬的一些數值分析限制。

基於上述原因，建立三維暫態水份流動模式及數值分析技術乃爲從事三維邊坡穩定分析之前提。因此，本文之目的即由建立非線性模式著手，並選擇最容易分析之顯式中央差分法 (Explicit Time with Central Space Difference Method) 用於解析該模式時之一致性 (consistency)、穩定性 (stability) 及收斂性 (convergency)，以作爲進一步設計研究用程式之依據。

主控方程式

三維坡地座標系統之選定在本文選用直角座標系統爲探討對象，原點設定於坡腳，z 軸向上爲正。且同一般的質量傳輸控容分析 (control volume analysis) (Geankoplis，1973) 方式，一單元體在 x 軸方向，單位時間內由左面流入之質量流出右面之質量分別爲：

$$\text{mass}_x = (A\cdot q\cdot\rho)_x$$

$$\text{mass}_{x+\Delta x} = (A\cdot q\cdot\rho)_x + \frac{\partial(A\cdot q\cdot\rho)}{\partial X}\cdot\Delta X \quad\text{.........(1)}$$

q 爲流束密度 (flux density)，因次爲 $L^3L^{-2}T^{-1}$ 。
ρ 爲流體密度 (fluid desity)，因次爲 ML^{-3} 。
A 爲面積，因次爲 L^{-2} 。

則 x 方向之淨質量流入率爲：

$$\text{mass}_x - \text{mass}_{x+\Delta x} = -\frac{\partial(A\cdot q\cdot\rho)_x}{\partial X}\cdot\Delta X \quad\text{..........(1a)}$$

同理 Y 及 Z 方向之淨質量流入率爲：

$$\text{mass}_y - \text{mass}_{y+\Delta y} = -\frac{\partial(A\cdot q\cdot\rho)_y}{\partial Y}\cdot\Delta Y \quad\text{..........(1b)}$$

$$\text{mass}_z - \text{mass}_{z+\Delta z} = -\frac{\partial(A\cdot q\cdot\rho)_z}{\partial Z}\cdot\Delta Z \quad\text{..........(1c)}$$

故單元體內之淨質量流入率爲：

$$-\left[\frac{\partial(A\cdot q\cdot\rho)_x}{\partial X}\cdot\Delta X + \frac{\partial(A\cdot q\cdot\rho)_y}{\partial Y}\cdot\Delta Y + \frac{\partial(A\cdot q\cdot\rho)_z}{\partial Z}\cdot\Delta Z\right] \quad\text{..........................(2)}$$

因爲 $A_x = \Delta Y\cdot\Delta Z$ ，$A_y = \Delta X\cdot\Delta Z$ ，$A_z = \Delta X\cdot\Delta Y$ ；所以 (2) 式改寫爲：

$$-\left[\frac{\partial(q\cdot\rho)_x}{\partial X} + \frac{\partial(q\cdot\rho)_y}{\partial Y} + \frac{\partial(q\cdot\rho)_z}{\partial Z}\right]\cdot\Delta X\,\Delta Y\,\Delta Z \quad\text{..........(2')}$$

(2') 之値應等於該單元質量隨時間的變化率：

$$\frac{\partial(E \cdot S \cdot \rho \cdot V)}{\partial t} \qquad (3)$$

其中 E 爲孔隙率 (porosity)。

S 爲飽和度 (saturation degree)。

V 爲體積，且等於 $\Delta X \Delta Y \Delta Z$。

因此

$$\frac{\partial(E \cdot S \cdot \rho)}{\partial t}\Delta X \Delta Y \Delta Z = -\left[\frac{\partial(q \cdot \rho)_x}{\partial X} + \frac{\partial(q \cdot \rho)_y}{\partial Y} + \frac{\partial(q \cdot \rho)_z}{\partial Z}\right]\Delta X \Delta Y \Delta Z \qquad (3')$$

即 $\frac{\partial(E \cdot S \cdot \rho)}{\partial t} = -\operatorname{div}(\rho \vec{q})$ ……(4)

第 (4) 式即爲連續方程式。

因 $\theta = E \cdot S$，θ 爲體積含水量，並考慮水體爲非壓縮性流體；ρ 等於常數。則簡化得到

$$\frac{\partial \theta}{\partial t} = -\operatorname{div}(\vec{q}) \qquad (4')$$

當達西定律 (Darcy slaw) 對於飽和及不飽和土壤均能適用時：

$$\vec{q} = -K(\theta) \cdot (\nabla \phi) \qquad (5)$$

$K(\theta)$ 爲水力導水度 (hydraulic conductivity)，並爲 θ 之函數。

ϕ 爲總勢能水頭 (total potential head)，因次爲L。

一般常溫條件下，ϕ 爲位能、壓力勢能與重能之總和，即

$$\phi = Z + \frac{1}{g}\int \frac{dp}{\rho} + \frac{\nu^2}{2g} \qquad (6)$$

由於土體中水流之速度水頭極小，可予以忽略。故僅得

$$\phi = Z + \frac{1}{g}\int \frac{dp}{\rho} = \varphi_g + \varphi_\rho$$

φ_g 稱爲重力水勢頭 (Gravitational potential head)

φ_ρ 稱爲壓力勢水頭 (Pressure potential head)

將 ϕ 代入連續方程式 (4') 及流束項 (5)，得到

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [K(\theta) \cdot (\nabla \phi)]$$

$$= \nabla \cdot \left[K(\theta)\frac{\partial[\varphi_g+\varphi_\rho]}{\partial X}\vec{i} + K(\theta)\frac{\partial[\varphi_g+\varphi_\rho]}{\partial Y}\vec{j} + K(\theta)\frac{\partial[\varphi_g+\varphi_\rho]}{\partial Z}\vec{k}\right]$$

$$= \left[\frac{\partial K(\theta)}{\partial X}\frac{\partial \varphi_\rho}{\partial X} + K(\theta)\frac{\partial^2 \varphi_\rho}{\partial X^2} + \frac{\partial K(\theta)}{\partial Y}\frac{\partial \varphi_\rho}{\partial Y} + K(\theta) + \frac{\partial^2 \varphi_\rho}{\partial Y^2} + \frac{\partial K(\theta)}{\partial Z}\left(1 + \frac{\partial \varphi_\rho}{\partial Z}\right) + K(\theta)\frac{\partial^2 \varphi_\rho}{\partial Z^2}\right] \qquad (7)$$

將第 (7) 式重新整理得：

$$\frac{\partial \theta}{\partial t} = K(\theta)\left(\frac{\partial^2 \varphi_\rho}{\partial X^2} + \frac{\partial^2 \varphi_\rho}{\partial Y^2} + \frac{\partial^2 \varphi_\rho}{\partial Z^2}\right) \frac{\partial K(\theta)}{\partial X}\frac{\partial \varphi_\rho}{\partial X} + \frac{\partial K(\theta)}{\partial Y}\frac{\partial \varphi_\rho}{\partial Y} + \frac{\partial K(\theta)}{\partial Z}\frac{\partial \varphi_\rho}{\partial Z} + \frac{\partial K(\theta)}{\partial Z} \qquad (7')$$

第 (7') 式即爲坡地三維暫態水份流動模式之主控方程式，此式與Lin (1998) 所得相同。其中 φ_ρ 在負荷下以 φ_m，即一般通稱的基勢能爲代表。

由於 θ 爲 φ_m 的函數；令 φ_m 值等於 H，則 (7') 式等於：

$$\frac{\partial \theta}{\partial t} = K(H)\left(\frac{\partial^2 H}{\partial X^2} + \frac{\partial^2 H}{\partial Y^2} + \frac{\partial^2 H}{\partial Z^2}\right) + \frac{\partial K(H)}{\partial X}\frac{\partial H}{\partial X} + \frac{\partial K(H)}{\partial Y}\frac{\partial H}{\partial Y} + \frac{\partial K(H)}{\partial Z}\frac{\partial H}{\partial Z} + \frac{\partial K(H)}{\partial Z} \qquad (8)$$

利用鏈鎖律 (chain rule) 的運算方法，可得：

$$\frac{\partial H}{\partial t}\frac{\partial \theta}{\partial H}=K(H)\left[\frac{\partial^2 H}{\partial X^2}+\frac{\partial^2 H}{\partial Y^2}+\frac{\partial^2 H}{\partial Z^2}\right]+\frac{\partial K(H)}{\partial X}\frac{\partial H}{\partial X}+\frac{\partial K(H)}{\partial Y}\frac{\partial H}{\partial Y}+\frac{\partial K(H)}{\partial Z}\frac{\partial H}{\partial Z}+\frac{\partial K(H)}{\partial Z} \quad \text{......(8')}$$

(8') 爲主控方程式之另一型式。

在解析主控方程式時，吾人引用 Gardner (1953) 所提出，且普遍應用於土壤水分特性之水力導水度 K 與毛細勢能 H 指數關係式

$$K(H)=K_s\exp(\alpha H) \quad \text{......(9)}$$

Ks 爲飽和導水度 (saturated conductivity)，因次爲 LT^{-1}。
α 爲經驗常數，其因次爲 L^{-1}；且 $\alpha>0$。

進一步推導 Ben — Asher et al. (1978) 之假設，得到體積含水量 θ 與基勢能 H 間之關係簡式爲

$$\theta=K_s\exp(\alpha H)/k \quad \text{......(10)}$$
$$=K(H)/k$$

k 爲經驗常數，其因次爲 LT^{-1}；且 $k>0$。

將 (9)、(10) 代入 (8') 可得

$$H_t=k/\alpha(H_{xx}+H_{yy}+H_{zz})+k(H^2{}_x+H^2{}_y+H^2{}_z)+kH_z \quad \text{......(11)}$$

有限差分數值分析

對於第 (11) 式之差分解，吾人採用顯式中央差分法 (Explicit Time with Central Space Difference Method) 處理；並且對此非線性偏微分方程式以其差分解 h 表達成如下通式

$$h_t=\Phi(t, X, Y, Z, h_x, h_y, h_z, h_{xx}, h_{yy}, h_{zz}) \quad \text{.....(12)}$$

近似爲

$$\frac{h^{r+1}_{i,j,k}-h^{r}_{i,j,k}}{\Delta t}\approx$$

$$\Phi\left(r\Delta t, i\Delta X, j\Delta Y, k\Delta Z, h^r_{i,j,k}, \frac{h^r_{i+1,j,k}-h^r_{i-1,j,k}}{2\Delta X}, \frac{h^r_{i,j+1,k}-h^r_{i,j-1,k}}{2\Delta Y}, \frac{h^r_{i,j,k+1}-h^r_{i,j,k-1}}{2\Delta Z}, \frac{h^r_{i+1,j,k}-2h^r_{i,j,k}+h^r_{i-1,j,k}}{\Delta X^2}, \frac{h^r_{i,j+1,k}-2h^r_{i,j,k}+h^r_{i,j-1,k}}{\Delta Y^2}, \frac{h^r_{i,j,k+1}-2h^r_{i,j,k}+h^r_{i,j,k-1}}{\Delta Z^2}\right) \quad \text{......(13)}$$

其中 r, i, j, k 分別爲對應於時間及 X, Y, Z 之指標。
上式爲 (12) 式之正合有限差分表示式 (exact finite difference expression)。

數值分析之重要工作乃在推導本文解析主控方程式 (8') 時，所用顯式中央差分法 (13) 式之穩定性 (stability)、一致性 (consistency) 及收斂性 (convergence) 分析。

一、穩定性分析

由於在數值計算中含有基本型態的誤差，諸如繼承誤差 (Inherent error)、截取誤差 (Truncation error) 及捨入誤差 (Roundoff error) 等。因此，限制誤差，使其不致傳播，乃爲數值分析技術之必要考慮條件。

對於 (13) 式之正合有限差分表示式，若將所有誤差一併考慮時，則寫成

$$\frac{(h^{r+1}_{i,j,k}+\varepsilon^{r}_{i+1,j,k})-(h^{r}_{i,j,k}+\varepsilon^{r}_{i,j,k})}{\Delta t}=$$

$$\Phi\left[t, X, Y, Z, h^r_{i,j,k}+\varepsilon^r_{i,j,k}, \frac{(h^r_{i+1,j,k}+\varepsilon^r_{i+1,j,k})-(h^r_{i-1,j,k}+\varepsilon^r_{i-1,j,k})}{2\Delta X},\right.$$

$$\frac{(h^r_{i,j+1,k}+\varepsilon^r_{i,j+1,k})-(h^r_{i,j-1,k}+\varepsilon^r_{i,j-1,k})}{2\Delta Y},$$

$$\frac{(h^r_{i,j,k+1}+\varepsilon^r_{i,j,k+1})-(h^r_{i,j,k-1}+\varepsilon^r_{i,j,k-1})}{2\Delta Z},$$

$$\frac{(h^r_{i+1,j,k}+\varepsilon^r_{i-1,j,k})-2(h^r_{i,j,k}+\varepsilon^r_{i,j,k})+(h^r_{i-1,j,k}+\varepsilon^r_{i-1,j,k})}{\Delta X^2},$$

$$\frac{(h^r_{i+1,j,k}+\varepsilon^r_{i+1,j,k})-2(h^r_{i,j,k}+\varepsilon^r_{i,j,k})+(h^r_{i-1,j,k}+\varepsilon^r_{i-1,j,k})}{\Delta Y^2},$$

$$\frac{(h^r_{i+1,j,k}+\varepsilon^r_{i+1,j,k})-2(h^r_{i,j,k}+\varepsilon^r_{i,j,k})+(h^r_{i-1,j,k}+\varepsilon^r_{i-1,j,k})}{\Delta Z^2}\Bigg]$$

....(14)

(14) 式重寫為

$$\frac{(h^{r+1}_{i,j,k}-h^r_{i,j,k})}{\Delta t}+\frac{(\varepsilon^{r+1}_{i,j,k}-\varepsilon^r_{i,j,k})}{\Delta t}=\Phi\Bigg[t,X,Y,Z,h^r_{i,j,k}+\varepsilon^r_{i,j,k},$$
$$\frac{(h^r_{i+1,j,k}+h^r_{i-1,j,k})}{2\Delta X}+\frac{(\varepsilon^r_{i+1,j,k}+\varepsilon^r_{i-1,j,k})}{2\Delta X},$$
$$\frac{(h^r_{i,j+1,k}+h^r_{i,j-1,k})}{2\Delta Y}+\frac{(\varepsilon^r_{i,j+1,k}+\varepsilon^r_{i,j-1,k})}{2\Delta Y},$$
$$\frac{(h^r_{i,j,k+1}+h^r_{i,j,k-1})}{2\Delta Z}+\frac{(\varepsilon^r_{i,j,k+1}+\varepsilon^r_{i,j,k-1})}{2\Delta Z},$$
$$\frac{(h^r_{i+1,j,k}-2h^r_{i,j,k}+h^r_{i-1,j,k})}{\Delta X^2},\frac{(\varepsilon^r_{i+1,j,k}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i-1,j,k})}{\Delta X^2},$$
$$\frac{(h^r_{i,j+1,k}-2h^r_{i,j,k}+h^r_{i,j-1,k})}{\Delta Y^2},\frac{(\varepsilon^r_{i,j+1,k}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i,j-1,k})}{\Delta Y^2},$$
$$\frac{(h^r_{i,j,k+1}-2h^r_{i,j,k}+h^r_{i,j,k-1})}{\Delta Z^2},\frac{(\varepsilon^r_{i,j,k+1}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i,j,k-1})}{\Delta Z^2}\Bigg]$$

....(14')

因為其中

$$\varepsilon^r_{i,j,k},\ \frac{(\varepsilon^r_{i+1,j,k}-\varepsilon^r_{i-1,j,k})}{2\Delta X},\ \frac{(\varepsilon^r_{i,j+1,k}-\varepsilon^r_{i,j-1,k})}{2\Delta Y},$$
$$\frac{(\varepsilon^r_{i,j,k+1}-\varepsilon^r_{i,j,k-1})}{2\Delta Z},\ \frac{(\varepsilon^r_{i+1,j,k}-2e^r_{i,j,k}+\varepsilon^r_{i-1,j,k})}{\Delta X^2},$$
$$\frac{(\varepsilon^r_{i,j+1,k}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i,j-1,k})}{\Delta Y^2},\ \frac{(\varepsilon^r_{i,j,k+k}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i,j,k-1})}{\Delta Z^2},$$

與

$$h^r_{i,j,k},\ \frac{(h^r_{i+1,j,k}-h^r_{i-1,j,k})}{2\Delta X},\ \frac{(h^r_{i,j+1,k}-h^r_{i,j-1,k})}{2\Delta Y},$$
$$\frac{(h^r_{i,j,k+1}-h^r_{i,j,k-1})}{2\Delta Z},\ \frac{(h^r_{i+1,j,k}-2h^r_{i,j,k}+h^r_{i-1,j,k})}{\Delta X^2},$$
$$\frac{(h^r_{i,j+1,k}-2h^r_{i,j,k}+h^r_{i,j-1,k})}{\Delta Y^2},\ \frac{(h^r_{i,j,k+1}-2h^r_{i,j,k}+h^r_{i,j,k-1})}{\Delta Z^2},$$

各項相較極微。故利用泰勒展開式 (Taylors expansion) 將函數Φ展開為

$$\frac{h^{r+1}_{i,j,k}-h^r_{i,j,k}}{\Delta t}+\frac{\varepsilon^{r+1}_{i,j,k}-\varepsilon^r_{i,j,k}}{\Delta t}=$$
$$\Phi\Big(t,X,Y,Z,h^r_{i,j,k},\frac{h^r_{i+1,j,k}-h^r_{i-1,j,k}}{2\Delta X},\frac{h^r_{i,j+1,k}-h^r_{i,j-1,k}}{2\Delta Y},$$
$$\frac{h^r_{i,j,k+1}-h^r_{i,j,k-1}}{2\Delta Z},\frac{h^r_{i+1,j,k}-2h^r_{i,j,k}+h^r_{i-1,j,k}}{\Delta X^2},$$
$$\frac{h^r_{i,j+1,k}-2h^r_{i,j,k}+h^r_{i,j-1,k}}{\Delta Y^2},\frac{h^r_{i,j,k+1}-2h^r_{i,j,k}+h^r_{i,j,k-1}}{\Delta Z^2}\Big)+$$
$$\frac{\partial\Phi}{\partial h}\Big|_{\xi 1}\varepsilon^r_{i,j,k}+\frac{\partial\Phi}{\partial h_x}\Big|_{\xi 2}\frac{\varepsilon^r_{i+1,j,k}-\varepsilon^r_{i-1,j,k}}{2\Delta X}+$$
$$\frac{\partial\Phi}{\partial h_y}\Big|_{\xi 3}\frac{\varepsilon^r_{i,j+1,k}-\varepsilon^r_{i,j-1,k}}{2\Delta Y}+\frac{\partial\Phi}{\partial h_z}\Big|_{\xi 4}\frac{\varepsilon^r_{i,j,k+1}-\varepsilon^r_{i,j,k-1}}{2\Delta Z}$$
$$+\frac{\partial\Phi}{\partial h_{xx}}\Big|_{\xi 5}\frac{(\varepsilon^r_{i+1,j,k}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i-1,j,k})}{\Delta X^2}$$
$$+\frac{\partial\Phi}{\partial h_{yy}}\Big|_{\xi 6}\frac{(\varepsilon^r_{i,j+1,k}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i,j-1,k})}{\Delta Y^2}$$
$$+\frac{\partial\Phi}{\partial h_{zz}}\Big|_{\xi 7}\frac{(\varepsilon^r_{i,j,k+1}-2\varepsilon^r_{i,j,k}+\varepsilon^r_{i,j,k-1})}{\Delta Z^2}$$

....(15)

其中

$$\xi 1=h^r_{i,j,k},\ \xi 2=\frac{h^r_{i+1,j,k}-h^r_{i-1,j,k}}{2\Delta X},\ \xi 3=\frac{h^r_{i,j+1,k}-h^r_{i,j-1,k}}{2\Delta Y},$$
$$\xi 4=\frac{h^r_{i,j,k+1}-h^r_{i,j,k-1}}{2\Delta Z},\ \xi 5=\frac{h^r_{i+1,j,k}-2h^r_{i,j,k}+h^r_{i-1,j,k}}{\Delta X^2},$$
$$\xi 6=\frac{h^r_{i,j+1,k}+2h^r_{i,j,k}+h^r_{i,j-1,k}}{\Delta Y^2},\ \xi 7=\frac{h^r_{i,j,k+1}-2h^r_{i,j,k}+h^r_{i,j,k-1}}{\Delta Z^2}$$

令下列係數成立

$$F_0=\left.\frac{\partial \Phi}{\partial h}\right|_{\xi 1},F_1=\left.\frac{\partial \Phi}{\partial h_x}\right|_{\xi 2},F_2=\left.\frac{\partial \Phi}{\partial h_y}\right|_{\xi 3},F_3=\left.\frac{\partial \Phi}{\partial h_z}\right|_{\xi 4}$$

$$F_5=\left.\frac{\partial \Phi}{\partial h_{xx}}\right|_{\xi 5},F_6=\left.\frac{\partial \Phi}{\partial h_{yy}}\right|_{\xi 6},F_7=\left.\frac{\partial \Phi}{\partial h_{zz}}\right|_{\xi 7}$$

則將各係數代入第 (15) 式可得

$$\frac{\varepsilon_{i,j,k}^{\tau+1}-\varepsilon_{i,j,k}^{\tau}}{\triangle t}=$$

$$F_0\,\varepsilon_{i,j,k}^{\tau}+F_1\frac{\varepsilon_{i+1,j,k}^{\tau}-\varepsilon_{i-1,j,k}^{\tau}}{2\triangle X}+F_2\frac{\varepsilon_{i,j+1,k}^{\tau}-\varepsilon_{i,j-1,k}^{\tau}}{2\triangle Y}+$$

$$F_3\frac{\varepsilon_{i,j,k+1}^{\tau}-\varepsilon_{i,j,k-1}^{\tau}}{2\triangle Z}+F_4\frac{(\varepsilon_{i+1,j,k}^{\tau}-2\varepsilon_{i,j,k}^{\tau}+\varepsilon_{i-1,j,k}^{\tau})}{\triangle X^2}+$$

$$F_5\frac{(\varepsilon_{i,j+1,k}^{\tau}-2\varepsilon_{i,j,k}^{\tau}+\varepsilon_{i,j-1,k}^{\tau})}{\triangle Y^2}+F_6\frac{(\varepsilon_{i,j,k+1}^{\tau}-2\varepsilon_{i,j,k}^{\tau}+\varepsilon_{i,j,k-1}^{\tau})}{\triangle Z^2}$$

....(16)

並將 (16) 式中各誤差項之係數歸納合併，則爲

$$\varepsilon_{i,j,k}^{\tau+1}=$$

$$(F_4\frac{\triangle t}{\triangle X^2}+F_1\frac{\triangle t}{2\triangle X})\varepsilon_{i+1,j,k}^{\tau}+(F_4\frac{\triangle t}{\triangle X^2}-F_1\frac{\triangle t}{2\triangle X})\varepsilon_{i-1,j,k}^{\tau}+$$

$$(F_5\frac{\triangle t}{\triangle Y^2}+F_2\frac{\triangle t}{2\triangle Y})\varepsilon_{i,j+1,k}^{\tau}+(F_5\frac{\triangle t}{\triangle Y^2}-F_2\frac{\triangle t}{2\triangle Y})\varepsilon_{i,j-1,k}^{\tau}+$$

$$(F_6\frac{\triangle t}{\triangle Z^2}+F_3\frac{\triangle t}{2\triangle Z})\varepsilon_{i,j,k+1}^{\tau}+(F_6\frac{\triangle t}{\triangle Y^2}-F_3\frac{\triangle t}{2\triangle Z})\varepsilon_{i,j,k-1}^{\tau}+$$

$$(1+\triangle tF_0-2F_4\frac{\triangle t}{\triangle X^2}-2F_5\frac{\triangle t}{\triangle Y^2}-2F_6\frac{\triangle t}{\triangle Z^2})\varepsilon_{i,j,k}^{\tau} \quad ....(17)$$

令 $\lambda=\frac{\triangle t}{\triangle X^2},\mu=\frac{\triangle t}{\triangle Y^2},\nu=\frac{\triangle t}{\triangle Z^2}$ 則(17)式改寫爲

$$\varepsilon_{i,j,k}^{\tau+1}=$$

$$\lambda(F_4+F_1\frac{\triangle X}{2})\varepsilon_{i+1,j,k}^{\tau}+\lambda(F_4-F_1\frac{\triangle X}{2})\varepsilon_{i-1,j,k}^{\tau}+$$

$$\mu(F_5+F_2\frac{\triangle Y}{2})\varepsilon_{i,j+1,k}^{\tau}+\mu(F_5-F_2\frac{\triangle Y}{2})\varepsilon_{i,j-1,k}^{\tau}+$$

$$\nu(F_6+F_3\frac{\triangle Z}{2})\varepsilon_{i,j,k+1}^{\tau}+\nu(F_6-F_3\frac{\triangle Z}{2})\varepsilon_{i,j,k-1}^{\tau}+$$

$$[1+\triangle tF_0-2(F_4\lambda+F_5\mu+F_6\nu)]\varepsilon_{i,j,k}^{\tau} \quad ....(17')$$

此時第 (17') 式右邊各誤差項之係數總和恰等於 1，故若能再使各項誤差擁有正係數，則本差分式便具有穩定 (Lapidus and Pinder, 1982) 因此，穩定條件爲：

$$\triangle X \leqq \left[\frac{2F_4}{|F_1|}\right]_{min} \quad ....(18a)$$

$$\triangle Y \leqq \left[\frac{2F_5}{|F_2|}\right]_{min} \quad ....(18b)$$

$$\triangle Z \leqq \left[\frac{2F_6}{|F_3|}\right]_{min} \quad ....(18c)$$

$$F_4\lambda+F_5\mu+F_6\nu \leqq (1+\triangle tF_0)/2 \quad ....(18d)$$

由於 $F_0=0$，$F_1=2kh_x$，$F_2=2kh_y$，$F_3=2kh_z$，$F_4=F_5=F_6=k/\alpha$。因此穩定條件成爲

$$\triangle X \leqq \left[\frac{1}{\alpha|h_x|}\right]_{min} \quad ....(19a)$$

$$\triangle Y \leqq \left[\frac{1}{\alpha|h_y|}\right]_{min} \quad ....(19b)$$

$$\triangle Z \leqq \left[\frac{1}{\alpha|h_z|}\right]_{min} \quad ....(19c)$$

$$\lambda+\mu+\nu \leqq \frac{\alpha}{2k} \quad ....(19d)$$

這些條件的設定，對於以後電腦程式的設計，有重要意義。

二、一致性分析

既已獲得本差分法所需之穩定條件，接續則在證得其一致性，以確定最終所解之微分方程式與主控方程式一致。由本文

所用之差分法表達第(11)式爲

$$\frac{h^{r+1}_{i,j,k}-h^{r}_{i,j,k}}{\Delta t}=\frac{k}{\alpha}\left(\frac{h^{r}_{i+1,j,k}-2h^{r}_{i,j,k}+h^{r}_{i-1,j,k}}{\Delta X^2}+\frac{h^{r}_{i,j+1,k}-2h^{r}_{i,j,k}+h^{r}_{i,j-1,k}}{\Delta Y^2}+\frac{h^{r}_{i,j,k+1}-2h^{r}_{i,j,k}+h^{r}_{i,j,k-1}}{\Delta Z^2}\right)+k\left[\left(\frac{h^{r}_{i-1,j,k}+h^{r}_{i+1,j,k}}{2\Delta X}\right)^2+\left(\frac{h^{r}_{i,j+1,k}+h^{r}_{i,j-1,k}}{2\Delta Y}\right)^2+\left(\frac{h^{r}_{i,j,k+1}+h^{r}_{i,j,k-1}}{2\Delta Z}\right)^2\right]+k\left(\frac{h^{r}_{i,j,k+1}+h^{r}_{i,j,k-1}}{2\Delta Z}\right)\quad\text{......(20)}$$

因爲

$h^{r+1}_{i,j,k}$，$h^{r}_{i+1,j,k}$，$h^{r}_{i-1,j,k}$，$h^{r}_{i,j+1,k}$，$h^{r}_{i,j-1,k}$，$h^{r}_{i,j,k+1}$，$h^{r}_{i,j,k-1}$

等各近似値之泰勒展開式爲

$$h^{r+1}_{i,j,k}=h^{r}_{i,j,k}+\Delta t\left.\frac{\partial h}{\partial t}\right|^{r}_{i,j,k}+\frac{\Delta t^2}{2!}\left.\frac{\partial^2 h}{\partial t^2}\right|^{r}_{i,j,k}+\frac{\Delta t^3}{3!}\left.\frac{\partial^3 h}{\partial t^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

$$h^{r}_{i+1,j,k}=h^{r}_{i,j,k}+\Delta X\left.\frac{\partial h}{\partial X}\right|^{r}_{i,j,k}+\frac{\Delta X^2}{2!}\left.\frac{\partial^2 h}{\partial X^2}\right|^{r}_{i,j,k}+\frac{\Delta X^3}{3!}\left.\frac{\partial^3 h}{\partial X^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

$$h^{r}_{i-1,j,k}=h^{r}_{i,j,k}-\Delta X\left.\frac{\partial h}{\partial X}\right|^{r}_{i,j,k}+\frac{\Delta X^2}{2!}\left.\frac{\partial^2 h}{\partial X^2}\right|^{r}_{i,j,k}-\frac{\Delta X^3}{3!}\left.\frac{\partial^3 h}{\partial X^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

$$h^{r}_{i,j+1,k}=h^{r}_{i,j,k}+\Delta Y\left.\frac{\partial h}{\partial Y}\right|^{r}_{i,j,k}+\frac{\Delta Y^2}{2!}\left.\frac{\partial^2 h}{\partial Y^2}\right|^{r}_{i,j,k}+\frac{\Delta X^3}{3!}\left.\frac{\partial^3 h}{\partial Y^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

$$h^{r}_{i,j-1,k}=h^{r}_{i,j,k}-\Delta Y\left.\frac{\partial h}{\partial Y}\right|^{r}_{i,j,k}+\frac{\Delta Y^2}{2!}\left.\frac{\partial^2 h}{\partial Y^2}\right|^{r}_{i,j,k}-\frac{\Delta Y^3}{3!}\left.\frac{\partial^3 h}{\partial Y^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

$$h^{r}_{i,j,k+1}=h^{r}_{i,j,k}+\Delta Z\left.\frac{\partial h}{\partial Z}\right|^{r}_{i,j,k}+\frac{\Delta Z^2}{2!}\left.\frac{\partial^2 h}{\partial Z^2}\right|^{r}_{i,j,k}+\frac{\Delta Z^3}{3!}\left.\frac{\partial^3 h}{\partial Z^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

$$h^{r}_{i,j,k-1}=h^{r}_{i,j,k}-\Delta Z\left.\frac{\partial h}{\partial Z}\right|^{r}_{i,j,k}+\frac{\Delta Z^2}{2!}\left.\frac{\partial^2 h}{\partial Z^2}\right|^{r}_{i,j,k}-\frac{\Delta Z^3}{3!}\left.\frac{\partial^3 h}{\partial Z^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

$$(h^{r}_{i+1,j,k}-h^{r}_{i-1,j,k})^2=4\Delta X^2\left(\frac{\partial h}{\partial X}\right)^2\Big|^{r}_{i,j,k}+4/3\,\Delta X^4+\left.\frac{\partial h\,\partial^3 h}{\partial X\,\partial X^3}\right|^{r}_{i,j,k}\ldots\ldots$$

$$(h^{r}_{i,j+1,k}-h^{r}_{i,j-1,k})^2=4\Delta Y^2\left(\frac{\partial h}{\partial Y}\right)^2\Big|^{r}_{i,j,k}+4/3\,\Delta Y^4+\left.\frac{\partial h\,\partial^3 h}{\partial Y\,\partial Y^3}\right|^{r}_{i,j,k}\ldots\ldots$$

$$(h^{r}_{i,j,k+1}-h^{r}_{i,j,k-1})^2=4\Delta Z^2\left(\frac{\partial h}{\partial Z}\right)^2\Big|^{r}_{i,j,k}+4/3\,\Delta Z^4+\left.\frac{\partial h\,\partial^3 h}{\partial Z\,\partial Z^3}\right|^{r}_{i,j,k}+\ldots\ldots$$

將上述各展開式代入第(20)式可得

$$\frac{\partial h}{\partial t}-\frac{k}{\alpha}\left(\frac{\partial^2 h}{\partial X^2}+\frac{\partial^2 h}{\partial Y^2}+\frac{\partial^2 h}{\partial Z^2}\right)-k\left[\left(\frac{\partial h}{\partial X}\right)^2+\left(\frac{\partial h}{\partial Y}\right)^2+\left(\frac{\partial h}{\partial Z}\right)^2\right]-k\frac{\partial h}{\partial Z}=-\frac{\Delta t}{2}\frac{\partial^2 h}{\partial t^2}-\frac{\Delta t^2}{6}\frac{\partial^3 h}{\partial t^3}+\frac{k}{\alpha}\left[\frac{\Delta X^2}{12}\frac{\partial^4 h}{\partial X^4}+\frac{\Delta Y^2}{12}\frac{\partial^4 h}{\partial Y^4}+\frac{\Delta Z^2}{12}\frac{\partial^4 h}{\partial Z^4}\right]^{r}_{i,j,k}+k\left[\frac{\Delta X^2}{3}\frac{\partial h}{\partial X}\frac{\partial^3 h}{\partial X^3}+\frac{\Delta Y^2}{3}\frac{\partial h}{\partial Y}\frac{\partial^3 h}{\partial Y^3}+\frac{\Delta Z^2}{3}\frac{\partial h}{\partial Z}\frac{\partial^3 h}{\partial Z^3}\right]^{r}_{i,j,k}+k\frac{\Delta Z^2}{6}\left.\frac{\partial^3 h}{\partial Z^3}\right|^{r}_{i,j,k}+O(\Delta t^3+\Delta X^3+\Delta Y^3+\Delta Z^3)\quad\text{......(21)}$$

當△t、△X、△Y、△Z趨近零時，第(21)式右邊各項亦逼近零。因此證明，第(21)式之差分解與主控方程式(11)具有一致性。

依據Lax(1961)之等値理論，一差分解若能達到穩定性及一致性，必能達到收斂性。因此，本文之差分解法，符合收斂條件。

以上之數値分析的穩定條件設定與分析方法之一致性的結論，對於後續之應用有其實用參考定義。相同的原理亦可延用於其他數値方法的解析。

結　論

三維暫態水份流動模式，採用Gardner所提出之水力導水度爲例，分析顯式中央差分法用於解析該模式時具一致性、穩定性及收斂性。但是，穩定條件亦顯示：倘若基勢能梯度太大，則條件變苛，對於網路區塊的設定將極爲細密，造成計算上的負擔。因此，本文所提示數値方法之適用性有其勢能梯度之上限制，其限制條件亦經解析，對於後續之程式設計具實用定義。

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