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A Study on the Vibration of an Elastic Sphere[1]

Li-Duhng Huarng[2] and Pictiaw Chen[3]

Abstract : A theoretical analysis of the vibration of an elastic sphere in spherical coordinates using the scalar and vector potential equations by the separation of variables method leads to the solutions of the trigonmetric equation, the Legendre equation and the Bessel equation. The associated Legendre polynomials of the first kind govern the mode shapes of vibration while the spherical Bessel functions of the first kind lead to the non-dimensional frequencies of the vibration modes The simplest vibration modes of a homogeneous elastic sphere are the pure compressional modes and the pure shear modes. The procedures to solve the non-dimensional frequencies for some cases of both the pure compressional modes and the pure shear modes are illustrated in the contents. Although, the solutions for the cases of the mixed modes of vibration are too complicated to be solved analytically, a different approach method published by Lamb in 1882 are included. A table for the non-dimensional frequencies of the mixed class vibration obtained numerically with a computer program for Poisson's ratio; ν; equals to 0.1, 0.2, 0.3 and 0.4, given by Cooke and Rand are also included for quick reference.

Key words : Vibration mode, Non-dimensional frequency, Elastic Sphere.

INTRODUCTION

The vibration of a solid elastic sphere is an old problem. The radibal oscillation of a sphere was studied as early as 1828 by S. D. Poisson [16]. The problem related to the elastic sphere was then continually developed by Lamb (1882) and other scienists. Lamb's work has the advantage of discussing the simpler modes of vibration of a uniform sphere in detsils with the method of obtaining roots of the frequency equations. He classified the general types of vibration into the "first class" and the "second class" [3,7,11,16]. The first class referred to the torsional class vibration modes which result in a dilation-free (no volume change) motion. The second class referred to the spheroidal vibration modes where the displacements have radial components and the volume may change.

The Navier equation which governs the dynamic responses of a linear, elastic, homogeneous, isotropic sphere can be dissloved into three equations, namely, the trigonometric

1. Contribution No. 1754 from Taiwan Agricultural Research Institute.
2. Respectively, Assistant Agricultural Engineer, Department of Agricultural Engineering, TARI, Wufeng, Taichung, Taiwan, ROC.
3. Professor, Biological and Agricultural Engineering Department, University of California, Davis, CA 95616—5294, USA.

equation, the Legendre equation and the Bessel equation[7,17]. Among the solutions to these equations, the Bessel functions of the first kind are used to obtain the non-dimensional frequencies which can be applied to predict the second and the third resonant frequencies of an intact apple from the first resonant frequencies while the Legendre polynomials are associated with the vibration modes[2,3,8].

MATERIALS AND METHODS

1. Theoretical model:

The Navier equation which governs the small motion of an isotropic elastic sphere without any body forces can be expressed as follows[1,2,13,14]:

$$\mu\nabla^2\bar{u}+(\lambda+\mu)\nabla[\nabla\cdot\bar{u}]=\rho\ddot{\bar{u}} \quad \cdots\cdots(1)$$

where:

λ, μ = Lame's constant

$\bar{u}$ = displacement vector

ρ = mass density

∇^2 = Laplacian operator

∇ = gradient operator

The solutions of equation (1) could be obtained if the displacement vector $\bar{u}$ is represented in terms of a scalar potential ϕ and a vector potential $\bar{\Psi}$[1,2,7,8]:

$$\bar{u}=\nabla\phi+\nabla\Psi \quad \cdots\cdots(2)$$

Substituting equation (2) into equation (1), results in:

$$\mu\nabla^2(\nabla\phi+\nabla\times\bar{\Psi})+(\lambda+\mu)\nabla[\nabla\cdot(\nabla\phi+\nabla\times\bar{\Psi})]=\rho\frac{\partial^2[\nabla\phi+\nabla\times\bar{\Psi}]}{\partial t^2} \quad \cdots\cdots(3)$$

But $\nabla\cdot\nabla\times\bar{\Psi}=0$

and $\nabla\cdot\nabla\phi=\nabla^2\phi \quad \cdots\cdots(4)$

Therefore, equation 3 becomes

$$\nabla[(\lambda+2\mu)\nabla^2\phi-\rho\ddot{\phi}]+\nabla\times(\mu\nabla^2\bar{\Psi}-\rho\ddot{\bar{\Psi}})=0 \quad \cdots\cdots(5)$$

The displacement equation 2 will satisfy the Navier equation, if

$$\nabla^2\phi=\frac{1}{V_l^2}\ddot{\phi} \quad \text{where} \quad V_l=\left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}} \quad \cdots\cdots(6)$$

and

$$\nabla^2\bar{\Psi}=\frac{1}{V_s^2}\ddot{\bar{\Psi}} \quad \text{where} \quad V_s=\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \quad \cdots\cdots(7)$$

V_l and V_s are called the compressional (or longitudinal) phase velocity and shear (or flexural) phase velocity[3], respectively.

The vector wave equation 7 is satisfied if the following scalar wave function is satisfied [14]:

$$\nabla^2\Psi+\frac{\omega^2}{V_s^2}\Psi=0 \quad \cdots\cdots(8)$$

Equations 6 and 8 are similar and can be solved by the separation of variables method [2,14,17]. The detailed procedures for determining these solutins in a spherical coordinate system can be found in Huarng (1992). The following is a description of the procedure for determining the solution ϕ of the scalar wave equation 6.

If we define ϕ as a product of a displacement function and a time function as follows:

$$\phi=\Phi(r,\ \theta,\ \delta)\ T(t) \quad \cdots\cdots(9)$$

where r, $\theta$and δ are spherical coordinates as shown in Fig. 1, and t is time.

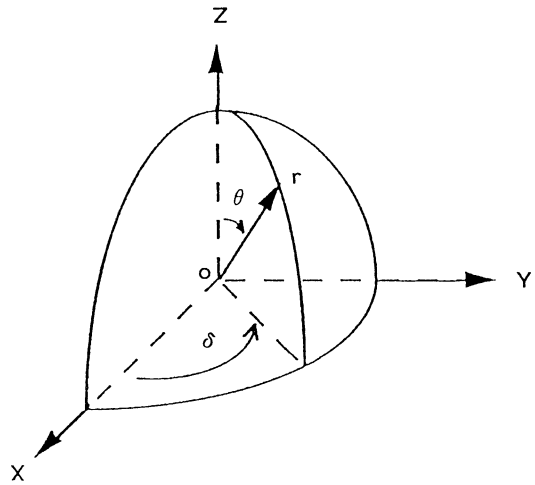


Fig. 1. Spherical coordinate for a sphere

Substituting eqtation 9 into equation 6 results in:

$$V_l^2\frac{\nabla^2\Phi}{\Phi}=\frac{\ddot{T}}{T} \quad \cdots\cdots(10)$$

Since the left hand side of equation 10 is a function of spatial variables only, and that on the right hand side is a function of time only, both sides must be equal to a constant. Therefore, we can write:

$$V_l^2\frac{\nabla^2\Phi}{\Phi}=\frac{\ddot{T}}{T}=-\omega^2 \quad \cdots\cdots(11)$$

Equation 11 yields

$$\ddot{T}+\omega^2T=0 \quad \cdots\cdots(12)$$

and

$$\nabla^2\Phi+\frac{\omega 2}{V_l^2}\Phi=0 \quad \cdots\cdots(13)$$

The solutions for equation 12 is of the form $T=e^{-i\omega t}$. Equaation 13 in spherical coordinates can be expressed as[7,9]:

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\ \frac{\partial^2\Phi}{\partial\delta^2}\right]+\frac{\omega^2}{V_l^2}\Phi=0 \quad \cdots\cdots(14)$$

Equation 14 is separable; thus by letting $\Phi(r, \theta, \delta) = R(r)\Theta(\theta)\triangle(\delta)$, one can separate equation 14 into the following three differential equations[7,17]:

$$\frac{d^2\triangle}{d\delta^2}+m^2\triangle=0 \quad \cdots\cdots(15)$$

$$\frac{d^2\Theta}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta}+\left(k^2-\frac{m^2}{\sin^2\theta}\right)\Theta=0 \quad \cdots\cdots(16)$$

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\frac{\omega^2}{V_t^2}-\frac{k^2}{r^2}\right)R=0 \quad \cdots\cdots(17)$$

where m^2 and k^2 are the separation constant.

The general solution of equation 15 is:

$$\triangle(\delta)=e^{-im\delta} \quad \cdots\cdots(18)$$

To obtain a single valued potential function with the range of δ to be $(0,2\pi)$, m must be an interer [4](i. e., $m=0,\pm1,\pm2,\pm3,\cdots\cdots$).

It can be shown that physically meaningful solutions of equation 16 exists only when $k^2=n(n+1)$, $n=0, 1, 2, 3,\cdots$., etc.[14,17]. For these values of k^2, the solutions of equation 16 are the associated Legendre polynomials of the first kind as represented below:

$$\Theta(\theta)=P_n^{|m|}(\cos\theta)$$

$$=(1-\cos^2\theta)^{\frac{|m|}{2}}\frac{d^{|m|}}{d\cos\theta^{|m|}}P_n(\cos\theta) \quad \cdots\cdots(19)$$

where $n=0, 1, 2,\cdots\cdots$; and $P_n(\cos\theta)$ is the Legendre polynomial which can be expressed in Rodrigues' formula form as[2,4,6,14]:

$$P_n(\cos\theta)=\frac{1}{2^n n!}\frac{d^n}{d(\cos\theta)^n}(\cos^2\theta-1)^n \quad \cdots\cdots(20)$$

Since $P_n(\cos\theta)$ is a polynomial of nth degree, $\Theta(\theta)$ vanishes when $m>n$. Therefore, the values of $|m|$ in equation 19 is limited to $|m| \leqq n$.

The solutions of equation 17 are the spherical Bessel functions of the first kind represented as:

$$R(r)=\sqrt{\frac{2}{\pi}\left(\frac{\omega}{V_t}\right)}Z_n\left(\frac{\omega r}{V_t}\right) \quad \cdots\cdots(21)$$

where

$$Z_n(\rho)=\sqrt{\frac{\pi}{2\rho}}Z_{n+1/2}(\rho) \quad \cdots\cdots(22)$$

and $Z_{n+1/2}(\rho)$ is the Bessel function of the first kind of order $n+1/2$.

Thereforefore, the solutions to equation 13 are:

$$\Phi_{mn}(r, \theta,\delta)=Z_n\left(\frac{\omega r}{V_t}\right)P_n^{|m|}(\cos\theta)\,e^{im\delta} \quad \cdots\cdots(23)$$

where $n=0, 1, 2, 3,\cdots\cdots$and $|m|=1, 2, \cdots\cdots, n$.

Combining the solutions of equations 12 and 13, one obtains the solutions of equation 6 as:

$$\phi_{nm}=Z_n\left(\frac{\omega r}{V_t}\right)P_n^{|m|}(\cos\theta)e^{im\delta}e^{-i\omega t} \quad \cdots\cdots(24)$$

It was shown by Pao and Mow (1973) that the vector wave equation

$$\nabla^2\bar{\Psi}+\frac{\omega^2}{V_s^2}\bar{\Psi}=0 \quad \cdots\cdots(25)$$

is satisfied if the following scalar wave function is satisfied. i. e.

$$\nabla^2\Psi+\frac{\omega^2}{V_s^2}\Psi=0 \quad \cdots\cdots(26)$$

The above scalar wave function can be solved in the same manner as used for the scalar function in equation (13). The result leads to the solutions of the form[2,7]

$$\Psi_{mn}(r,\ \theta,\ \delta)=Z_n\left(\frac{\omega r}{V_s}\right)\ P_n^{|m|}\ (\cos\theta)e^{|m\delta} \quad \cdots\cdots(27)$$

2. Vibration modes:

The torsional class vibration can be expressed as $T_{n\ell-1}$ and the spheroidal class vibration as $S_{n\ell-1}$[2], where n is a non-negative integer and $\ell=1, 2, 3,\cdots$ is the order of roots satisfying the frequency equations.

The simplest vibration modes of a homogeneous elastic sphere are pure compressional modes and pure shear modes. For the pure compressional modes, the procedure for obtaining the non-dimensional frequencies are illustated below:

2.1 Pure Compressional Modes

For a solid sppherical resonator, the spherical Bessel functions of the first kind can be applied. And the scalar potential would be

$$\Phi_{mn}(r,\ \theta,\ \delta)=Z_n\left(\frac{\omega r}{V_l}\right)\ P_n^{|m|}(\cos\theta)\ e^{|m\delta} \quad \cdots\cdots(23)$$

If m=n=0, from equation 2, these solutions simplify to

$$\begin{aligned}\bar{u}_{mn}&=\bar{u}_{00}\\&=\nabla\Phi_{00}\ (r,\ \theta,\ \delta)\\&=\nabla z_0\left(\frac{\omega r}{V_l}\right)\\&=\hat{r}\ \frac{d}{dr}\ z_0\left(\frac{\omega r}{V_l}\right)\end{aligned} \quad \cdots\cdots(28)$$

Where $\hat{r}$ is the unit position vector.

And the strain field associated with equation (28) has only 3 components[28]. Namely,

$$\begin{aligned}S_{rr}&=\frac{1}{i\omega}\ \frac{\partial u_r}{\partial r}\\S_{\theta\theta}&=\frac{1}{i\omega}\ \frac{u_r}{r}\\S_{\delta\delta}&=\frac{1}{i\omega}\ \frac{u_r}{r}\end{aligned} \quad \cdots\cdots(29)$$

where $i=\sqrt{-1}$.

Therefore, the traction force on the spherical surface is [2]

$$\begin{aligned}T.\ \hat{r}&=T_{rr}\\&=c_{11}\ S_{rr}+c_{12}\ (S_{\theta\theta}+S_{\delta\delta})\end{aligned} \quad \cdots\cdots(30)$$

So, the stress-free boundary condition on the spherical surface (r=a) is

$$c_{11}\frac{d^2}{dr^2}Z_0\left(\frac{\omega r}{V_t}\right)+\frac{2c_{12}}{r}\frac{d}{dr}Z_0\left(\frac{\omega r}{V_t}\right)=0 \quad \cdots\cdots(31)$$

Since $Z_0(\rho)=\sin\rho/\rho$[2,7,18], we know that

$$Z_0\left(\frac{\omega r}{V_t}\right)=\frac{\sin\left(\frac{\omega r}{V_t}\right)}{\left(\frac{\omega r}{V_t}\right)} \quad \cdots\cdots(32)$$

The definition of Poison's ratio, ν, in terms of the stiffness constants, c_{jn}, where j and n are integers, is as follows[2,7]:

$$\nu=\frac{c_{11}-2c_{44}}{2(c_{11}-c_{44})} \quad \cdots\cdots(33)$$

Therefore, the ratio of c_{11} to c_{44} can be found as:

$$\frac{c_{11}}{c_{44}}=\frac{2-2\nu}{1-2\nu} \quad \cdots\cdots(34)$$

and since $c_{12}=c_{11}-2c_{44}$ for isotropic materials[2,7], substitution of equation (32) into equation (31) at r=a yields the characteristic frequency equation:

$$\tan\left(\frac{\omega a}{V_t}\right)=\frac{\left(\frac{\omega a}{V_t}\right)}{1-\frac{c_{11}}{4c_{44}}\left(\frac{\omega a}{V_t}\right)^2}=\frac{\left(\frac{\omega a}{V_t}\right)}{1-\left(\frac{1-\nu}{2-4\nu}\right)\left(\frac{\omega a}{V_t}\right)^2} \quad \cdots\cdots(35)$$

If the roots of equation (35) are designated as s_{00l}, where the first two subscripts indicate m =n=0 in equation (23), the natural frequencies of these family of modes are

$$f=\frac{V_t}{2\pi a}s_{00l} \quad \cdots\cdots(36)$$

Where these modes are conventionally labeled S_{0l}.

Now, for the case of v=0.3, which is commonly used as Poisson's ratio for apples[3], equation (35) becomes

$$\tan\left(\frac{\omega a}{V_t}\right)=\frac{\left(\frac{\omega a}{V_t}\right)}{1-\left(\frac{7}{8}\right)\left(\frac{\omega a}{V_t}\right)^2} \quad \cdots\cdots(37)$$

The solutions to equation (37) are the non-dimensional frequencies, $\frac{\omega a}{V_t}$, designated as H (=ha). The first four roots (excluding the trivial solution, 0) of this equation are

$$H=ha=\frac{\omega a}{V_t}=2.6702,\ 6.0920,\ 9.3009,\ 12.4743 \quad \cdots\cdots(38)$$

which correspond to the S_{01}, S_{02}, S_{03} and S_{04} modes, respectively.

2.2 Pure shear modes

The particle motion under pure shear modes is always normal to the radial coordinate, and

$$u_\theta=\frac{1}{\sin\theta}\frac{\partial\Psi_{mn}}{\partial\delta} \quad \cdots\cdots(39)$$

$$u_\delta=-\frac{\partial\Psi_{mn}}{\partial\theta} \quad \cdots\cdots(40)$$

Because the field must remain finite at r=0, the spherical Bessel function of the first kind was considered in equation (27).

The strain field corresponding to equations (39) and (40) has 5 components, $S_{\theta\theta}$, $S_{\delta\delta}$, $S_{r\theta}$, $S_{r\delta}$ and $S_{\theta\delta}$[2,7]. Also the spherical surface traction force is

$$T.\hat{f}=\hat{\theta}\frac{c_{44}}{i\omega}\left(\frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)+\hat{\delta}\frac{c_{44}}{i\omega}\left(\frac{\partial u_\delta}{\partial r}-\frac{u_\delta}{r}\right) \quad \cdots\cdots(41)$$

where $\hat{\theta}$ and $\hat{\delta}$ are the spherical unit vector.

Substituting equations (39) and (40) into stress-free boundary condition $T.\hat{f}=0$ at r=a yields characteristic equation in the variable $\left(\frac{\omega a}{Vs}\right)$. Shear modes are conventionally designated as $T_{n\ell-1}$, where ℓ is the order of roots satisfying the frequency equation. For the case where m=0 and n=1, the particle velocity field is

$$\bar{u}_{01}=\hat{\delta}\frac{\partial}{\partial\theta}\Psi_{01}=-\hat{\delta}\cos\theta\ Z_1\left(\frac{\omega r}{V_s}\right) \quad \cdots\cdots(42)$$

and the frequency relation reduces to

$$\tan\left(\frac{\omega a}{V_s}\right)=\frac{\left(\frac{\omega a}{V_s}\right)}{1-\frac{1}{3}\left(\frac{\omega a}{V_s}\right)^2} \quad \cdots\cdots(43)$$

The solutions to equation (43) are the non-dimensional frequencies $\frac{\omega a}{V_s}$, designated as H (=ka). The first four roots (excluding the trivial solution, 0) of this equation are

$$H=ka=\frac{\omega a}{V_s}=5.7635,\ 9.0950,\ 12.3229,\ 15.5146 \quad \cdots\cdots(44)$$

which correspond to the T_{11}, T_{12}, T_{13} and T_{14} modes, respectively.

2.3 Mixed Modes

Imposition of the stress free boundary conditions at the sphere surface (r=a) to solve a combination of spherical shear wave equations and compressional equations leads to a complicated frequency relation, which must be solved numerically[2,15]. The modes associated with the complicated frequency relation are usually designated as $S_{n\ell-1}$.

The following is a different approach which was first published by Lamb in 1882. Before investigating these vibration frequencies, the $\Psi_n(\theta)$ function used in the mixed mode frequency equations is defined as[18].

$$\Psi_n(\theta)=\frac{\Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{1}{2}\theta\right)^{n+\frac{1}{2}}}Z_{n+\frac{1}{2}}(\theta)$$

$$=1-\frac{\theta^2}{2\cdot(2n+3)}+\frac{\theta_4}{2\cdot4\cdot(2n+3)\cdot(2n+5)}-\text{etc}\cdots\cdots$$

$$=(-1)^n\cdot1\cdot3\cdot5\cdots(2n+1)\cdot\left(\frac{d}{\theta\ d\theta}\right)^n\frac{\sin\theta}{\theta} \quad \cdots\cdots(45)$$

where $Z_{n+\frac{1}{2}}(\theta)$ is the Bessel equation and the Gamma function $\Gamma(\alpha)$ is defined as[9]:

$$\Gamma(\alpha)=\int_0^\infty e^{-t}\,t^{\alpha-1}\,dt \qquad (\alpha>0) \quad \cdots\cdots(46)$$

with the following properties:

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha) \quad \cdots\cdots(47)$$

and

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \quad \cdots\cdots(48)$$

The function $\Psi_n(\theta)$ has the following properties[10,11]:

$$\Psi_n'(\theta)=-\frac{\theta}{2n+3}\Psi_{n+1}(\theta) \quad \cdots\cdots(49)$$

$$\Psi_n(\theta)-\Psi_{n-1}(\theta)=\frac{\theta^2}{(2n+1)(2n+3)}\Psi_{n+1}(\theta) \quad \cdots\cdots(50)$$

which is useful in solving frequency equations and checking the values of the $\Psi_n(\theta)$. The first five $\Psi_n(\theta)$ function values are:

$$\begin{aligned}\Psi_1(\theta)&=(-1)^1\cdot1\cdot3\cdot\left(\frac{d}{\theta\, d\theta}\right)^1\frac{\sin\theta}{\theta}\\&=-3\left(\frac{1}{\theta}\right)\frac{d}{d\theta}\left(\frac{\sin\theta}{\theta}\right)\\&=\frac{-3\theta\cos\theta+3\sin\theta}{\theta^3}\end{aligned} \quad \cdots\cdots(51)$$

$$\begin{aligned}\Psi_2(\theta)&=(-1)^2\cdot1\cdot3\cdot5\cdot\left(\frac{d}{\theta\, d\theta}\right)^2\frac{\sin\theta}{\theta}\\&=15\left(\frac{1}{\theta}\frac{d}{d\theta}\right)\left(\frac{1}{\theta}\frac{d}{d\theta}\left(\frac{\sin\theta}{\theta}\right)\right)\\&=15\left(\frac{1}{\theta}\frac{d}{d\theta}\right)\left(\frac{1}{\theta}\left(\frac{\theta\cos\theta-\sin\theta}{\theta^2}\right)\right)\\&=15\left(\frac{1}{\theta}\frac{d}{d\theta}\right)\left(\frac{\theta\cos\theta-\sin\theta}{\theta^3}\right)\right)\\&=15\left(\frac{1}{\theta}\frac{3\theta\sin\theta-3\theta\cos\theta-\theta^2\sin\theta}{\theta^4}\right)\\&=\frac{-15\theta^2\sin\theta-45\theta\cos\theta+45\sin\theta}{\theta^5}\end{aligned} \quad \cdots\cdots(52)$$

$$\begin{aligned}\Psi_3(\theta)&=(-1)^3\cdot1\cdot3\cdot5\cdot7\cdot\left(\frac{d}{\theta\, d\theta}\right)^3\frac{\sin\theta}{\theta}\\&=-105\left(\frac{d}{\theta\, d\theta}\right)\left(\frac{d}{\theta\, d\theta}\right)\left(\frac{d}{\theta\, d\theta}\right)\frac{\sin\theta}{\theta}\\&=\frac{105\theta^3\cos\theta-630\theta^2\sin\theta-1575\theta\cos\theta+1575\sin\theta}{\theta^7}\end{aligned} \quad \cdots\cdots(53)$$

$$\begin{aligned}\Psi_4(\theta)&=(-1)^4\cdot1\cdot3\cdot5\cdot7\cdot9\cdot\left(\frac{d}{\theta\, d\theta}\right)^4\frac{\sin\theta}{\theta}\\&=945\left(\frac{d}{\theta\, d\theta}\right)\left(\frac{d}{\theta\, d\theta}\right)\left(\frac{d}{\theta\, d\theta}\right)\left(\frac{d}{\theta\, d\theta}\right)\frac{\sin\theta}{\theta}\\&=\frac{1}{\theta^9}(945\theta^4\sin\theta+9450\theta^3\cos\theta-42525\theta^2\sin\theta-99225\theta\cos\theta+99225\sin\theta)\end{aligned} \quad \cdots\cdots(54)$$

$$\Psi_5(\theta)=(-1)^5\cdot1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot\left(\frac{d}{\theta\, d\theta}\right)^5\frac{\sin\theta}{\theta}$$

$$=-10395\left(\frac{d}{\theta\, d\theta}\right)^5\frac{\sin\theta}{\theta}$$

$$=\frac{1}{\theta^{11}}(-10395\theta^5\cos\theta+155925\theta^4\sin\theta+1091475\theta^3\cos\theta$$

$$-4365900\theta^2\sin\theta-9823275\theta\cos\theta+9823275\sin\theta) \quad \cdots\cdots(55)$$

The compressional phase velocity V_l is also defined as[5]

$$V_l^2=\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)} \quad \cdots\cdots(56)$$

Now, since

$$ha=\frac{\omega a}{V_l} \quad \text{and} \quad ka=\frac{\omega a}{V_s} \quad \cdots\cdots(57)$$

Then since Lame's constants[11]

$$\mu=G \quad \text{and} \quad \lambda^2=\frac{h^2}{k_2} \quad \cdots\cdots(58)$$

From equations (56), (57), (58) and (7), we have

$$\lambda^2=h^2/k^2$$

$$=V_s^2/V_l^2$$

$$=\frac{G/\rho}{E(1-\nu)/\rho(1+\nu)(1-2\nu)}$$

$$=\frac{0.5-\nu}{1-\nu} \quad \cdots\cdots(59)$$

where ω is the angular velocity of every particle in harmonic motion and ρ is the density of the sphere.

For the mixed mode of vibration, the stress free boundary condition on the spherical surface (r=a) requires that[11,12,16]

$$A_n\omega_n+C_n\phi_n=0 \quad \cdots\cdots(60)$$

and

$$B_n\omega_n+D_n\phi_n=0 \quad \cdots\cdots(61)$$

where ω_n stands for a spherical solid harmonic of positive degree n, and ϕ_n is a solid harmonic of positive degree n. The coefficients A_n, B_n, C_n and D_n are given as follows[11,12,16]:

$$A_n=\frac{1}{h^2}\left(\frac{k^2a^2}{2n+1}\Psi_n(ha)-2(n-1)\Psi_{n-1}(ha)\right) \quad \cdots\cdots(62)$$

$$B_n=-\frac{k^2a^{2n+3}}{h^2(2n+1)}\left(\Psi_n(ha)+\frac{2(n+2)ha}{k^2a^2}\Psi_n'(ha)\right) \quad \cdots\cdots(63)$$

$$C_n=-\left(\frac{k^2a^2}{2n+1}\Psi_n(ka)-2(n-1)\Psi_{n-1}(ka)\right) \quad \cdots\cdots(64)$$

$$D_n=-\frac{nk^2a^{2n+3}}{(n+1)(2n+1)}\left(\Psi_n(ka)+\frac{2(n+2)}{ka}\Psi_n'(ka)\right) \quad \cdots\cdots(65)$$

2.3.1 Case n=2 ($\nu<0.5$)

Considering n=2, the equations (60) and (61) become

$$\left(\frac{k^2a^2}{5}\Psi_2(ha)-2\Psi_1(ha)\right)\frac{\omega_2}{h^2}-\left(\frac{k^2a^2}{5}\Psi_2(ka)-2\Psi_1(ka)\right)\Phi_2=0 \quad \cdots\cdots(66)$$

and

$$\left(\Psi_2(ha)+\frac{8ha}{k^2a^2}\Psi_2'(ka)\right)\frac{\omega_2}{h_2}+\frac{2}{3}\left(\Psi_2(ka)+\frac{8}{ka}\Psi_2'(ka)\right)\phi_2=0 \quad \cdots\cdots(67)$$

By eliminating the factor ω_2/ϕ_2 in equations (66) and (67) and expanding the results in rising power of H (=ka), the equation assumes the form

$$a_1-a_2H^2+a_3H^4-a_4H^6+a_5H^8-+\cdots=0 \quad \cdots\cdots(68)$$

The values of the coefficients are:

$$a_1=19-24\lambda^2 \quad \cdots\cdots(69)$$

$$a_2=\frac{37}{9}-\frac{7}{2}\lambda^2-\frac{4}{3}\lambda^4 \quad \cdots\cdots(70)$$

$$a_3=\frac{193}{792}+\frac{5}{126}\lambda^2-\frac{13}{56}\lambda^4-\frac{1}{33}\lambda^6 \quad \cdots\cdots(71)$$

$$a_4=\frac{25}{3861}+\frac{125}{11088}\lambda^2-\frac{1}{168}\lambda^4-\frac{185}{33264}\lambda^6-\frac{1}{2574}\lambda^8 \quad \cdots\cdots(72)$$

$$a_5=\frac{719}{7413120}+\frac{163}{432432}\lambda^2+\frac{19}{133056}\lambda^4-\frac{59}{299376}\lambda^6-\frac{253}{3459456}\lambda^8-\frac{1}{308880}\lambda^{10} \quad \cdots\cdots(73)$$

Substituting the relation between λ and ν from equation (59) for the case of $\nu=0.3$, equation (68) becomes

$$12.143-3.002H^2+0.235H^4-0.009H^6+0.0002H^8-+\cdots=0 \quad \cdots\cdots(74)$$

The first root of the frequency equation is H=2.6462, which corresponds to the simplest mode of this class, S_{20}. It is usually called oblate-prolate mode since the sphere is distorted into an ellipsoid and alternately becomes prolate and oblate according to the phase of the motion. Different ν value results in slightly different dimensionless frequency values.

2.3.3 Case n=3 ($\nu<0.5$)

Considering n=3, equations (60) and (61) become

$$\left(\frac{k^2a^2}{7}\Psi_3(ha)-4\Psi_2(ha)\right)\frac{\omega_3}{h^2}-\left(\frac{k^2a^2}{7}\Psi_3(ka)-4\Psi_2(ka)\right)\phi_3=0 \quad \cdots\cdots(75)$$

$$\left(\Psi_3(ka)+\frac{10ha}{k^2a^2}\Psi_3\cdot(ha)\right)\frac{\omega_3}{h^2}+\frac{3}{4}\left(\Psi_3(ka)+\frac{10}{ka}\Psi_3'(ka)\right)\phi_3=0 \quad \cdots\cdots(76)$$

Eliminating ω_3/ϕ_3 in equations (75) and (76), and substituting the function values of Ψ_2, Ψ_3 and Ψ_3', one will get a complicated expressions of sine and cosine functions. The numerical solutions of the various roots for $\nu=0.1$, 0.2, 0.3 and 0.4, given by Cooke and Rand, are listed in Table 1.

2.3.3 Case n=4 ($\nu<0.5$)

Considering n=4, equations (60) and (61) become

$$\left(\frac{k^2a^2}{9}\Psi_4(ha)-6\Psi_3(ha)\right)\frac{\omega_4}{h^2}-\left(\frac{k^2a^2}{9}\Psi_4(ka)-6\Psi_3(ka)\right)\phi_4=0 \quad \cdots\cdots(77)$$

$$\left(\Psi_4(ha)+\frac{12ha}{k^2a^2}\Psi_4'(ha)\right)\frac{\omega_4}{h^2}+\frac{4}{5}\left(\Psi_4(ka)+\frac{12}{ka}\Psi_4'(ka)\right)\phi_4=0 \quad \cdots\cdots(78)$$

Elimination of ω_4/ϕ_4 in equations (77) and (78) and substitution of the function values of Ψ_3, Ψ_4, and Ψ_4', results in a complicated expressions of sine and cosine functions.

The numerical solutions of the various roots for the different Poisson's rations, $\nu=0.1$, 0.2, 0.3 and 0.4, given by Cooke and Rand, are also listed in Table 1.

Table 1. Mixed class non-dimensional free vibration frequencies of a homogeneous elastic sphere, $H_{n\ell-1}$

$\ell=$	$\nu=$	n=2	n=3	n=4
		H_{20}	H_{30}	H_{40}
1	0.1	2.6152	3.8360	4.8747
	0.2	2.6327	3.8927	4.9689
	0.3	2.6462	3.9374	5.0460
	0.4	2.6569	3.9731	5.1092
		H_{21}	H_{31}	H_{41}
2	0.1	4.4129	5.9597	7.5475
	0.2	4.7171	6.2920	7.8959
	0.3	5.0070	6.6086	8.2147
	0.4	5.2619	6.8811	8.4745
		H_{22}	H_{32}	H_{42}
3	0.1	7.6340	9.1069	10.4555
	0.2	8.0884	9.5024	10.8375
	0.3	8.5272	9.8824	11.2250
	0.4	8.7591	10.1362	11.5163
		H_{23}	H_{33}	H_{43}
4	0.1	9.1699	10.8406	12.4908
	0.2	9.4539	11.3017	13.0631
	0.3	10.3759	12.4277	14.1626
	0.4	12.0138	13.4255	14.8026

RESULTS AND DISCUSSION

The Navier equation can be applied to simulate the vibration of a homogeneous elastic sphere. By using the scalar and vector potential to represent the displacement vector, one can obtain three equations, namely, the Legendre equation, the Bessel's equation and the trigonometric equation. The analytical soltuions to the Legendre equation are called the associated Legendre polynomials of the first kind, which governs the mode shapes of the vibration. The analytical solutions to the Bessel equation are the spherical Bessel functions of the first kind, which can be applied to solve the non-dimensional frequencies of either pure compressional modes, pure shear modes or mixed modes either analytically or numerically. The solutions to the trigonometric equation are the combination of sinusoidal and cosinusoidal functions.

The product of the trigonometric functions and the associated Legendre polyomials, $\sin(m\delta)\,P_n^{|m|}(\cos\theta)$ and $\cos(m\delta)\,P_n^{|m|}(\cos\theta)$, are known as tesseral harmonics of nth degree and $|m|$th order[17]. They are periodic functions on the surface of a unit sphere and the indices $|m|$ and n determine the number of nodal lines. For instance, m=0, the field is independent of the equatorial angle δ. If n is also zero, the function value is constant on the surface of the sphere. When n=1, the only nodal line is at the equator $\theta=90°$along which the function is zero. When n=2, there are two nodal lines parallel to the equator at $\theta=54.7°$and

$\theta=125.3°$, and the sphere is divided into three zones[7,17]. The function is negative over the equatorial zone and positive over the polar zones. It is clear that there are n nodal lines that divide the sphere into n+1 zones within which the function value is alternately positive and negative. If $m\neq0$, the number of nodal lines parallet to the equator is $n-|m|$ and the function is zero at the poles due to the factor $\sin^{|m|}\theta$ in the associated Legendre equation.

The vibration modes of the first three resonant frequencies of an intact apple induced by striking the equator of the fruit were proved to be an S_{20}, S_{30}, and S_{40}, respectively[7,8]. If we assume that apples are elastic sphere with Poisson's ratio, $\nu=0.3$[3], from Table 1, the non-dimensional frequencies associated with these three modes are 2.6462, 3.9374 and 5.0460, respectively. These values can be applied to conjecture the second and the third resonant frequencies from the first resonant frequencies by the ratios of the non-dimensional frequencies. If one uses the first resonant frequencies to calculate the corresponding second and third resonant frequencies, the coefficient of determination (R^2) between the measured and the calculated second resonant frequencies was 0.978 for the Red Delicious apples and 0.967 for the Golden Delicious apples, while R^2 between the measured and the calculated third resonant frequencies was 0.964 for the Red Delicious apples and 0.960 for the Golden Delicious apples[7]. Furthermore, the slope of every regression line between the measured and the calculated second or third resonant frequencies was statistically proven to be unity [7]. The comparison between these theoretical and experimental results will be extensively demonstrated in the next article.

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彈性球體振動之研究[1]

黃禮棟[2] 陳必超[3]

摘 要

在球體座標系中，利用常量及向量方程式代入之分離變數法於理論上分析彈性球體之振動可以得到三角幾何方程式、雷詹德方程式及白賽爾方程式等三種方程式。其中之第一類雷詹德聯合方程式主宰振動模式，而第一類球體座標白賽爾方程式則可導出該振動模式之無次元頻率。一個均質彈性球體最簡單之振動模式爲純壓縮模式及純剪切模式。本文中闡釋純壓縮模式及純剪切模式中在某些情況與條件下之無次元頻率之解析方法。雖然各種混合振動模式之無次元頻率之解答因爲過於複雜以致於無法使用理論分析而必須借重電子計算機以數值方法計算以求取近似值，但文中仍然列入了由藍姆（Lamb）於1882年所發表之解析方法以及庫克（Cooke）與藍德（Rand）二人利用電子計算機以數值分析法計算所得到溥松比（Poisson's ratio）之值分別爲0.1，0.2，0.3及0.4等條件下之無次元頻率的數值供值參考。

關鍵詞：振動模式、無次元頻率、彈性球體。

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1. 臺灣省農業試驗所 研究報告第 1754 號。
2. 本所農業工程系助理研究員。臺灣省 臺中縣 霧峰鄉。
3. 美國加利福尼亞州戴維斯加州大學生物與農業工程學系教授。美國 加州 戴維斯。